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Your Roll No...............

Sr. No. of Question Paper : 6624 **HC**

Unique Paper Code : 32351302

Name of the Paper : Group Theory 1

Name of the Course : **B.Sc. (Hons.) Mathematics**

Semester : III

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immec this question paper.
2. Attempt any **two** parts from each ques
3. **All** questions are compulsory.

1. (a) For a fixed point (a, b) in $\mathbb{R}^2$, define $T_{(a,b)} : \mathbb{R}^2 \to \mathbb{R}^2$ by $(x, y) \to (x + a, y + b)$.

   Show that $T(\mathbb{R}^2) = \{T_{a,b} \mid a, b \in \mathbb{R}\}$

   is a group under function composition. (6)

   (b) (i) Find the inverse of $\begin{pmatrix} 2 & 6 \\ 3 & 5 \end{pmatrix}$ in $GL(2, \mathbb{Z}_{11})$. (4)

(ii) Let G be an Abelian group under multiplication with identity e. Show that

$H = \{x^2 \mid x \in G\}$ is a subgroup of G. (2)

(c) (i) Let G be a group. Show that $Z(G) = \bigcap_{a \in G} C(a)$ where Z(G) is the Center of G and C(a) is the Centralizer of a. (4)

(ii) Let G be the group of nonzero real numbers under multiplication. Show that

$H = \{x \in G \mid x = 1 \text{ or } x \text{ is irrational}\}$

and $K = \{x \in G \mid x \geq 1\}$ are not subgroups of G. (2)

2. (a) Let G be a group and let $a \in G$. If $|a| = n$, prove that $\langle a \rangle = \{e, a, a^2, \ldots, a^{n-1}\}$ and $a^i = a^j$ if and only if n divides $i - j$. (6)

(b) Suppose that $|a| = 24$. Find a generator for $\langle a^{21} \rangle \cap \langle a^{10} \rangle$. (6)

(c) If $|a| = n$, show that

$$\langle a^k \rangle = \langle a^{\gcd(n,k)} \rangle$$

and that $$|a^k| = \frac{n}{\gcd(n,k)}$$ (6)

3. (a) Define the Alternating Group A_n. Show that it forms a subgroup of the Permutation Group S_n and $|A_n| = \frac{n!}{2}$. (6)

(b) Prove that every group is isomorphic to a group of permutations. (6)

(c) Prove that U(10) is not isomorphic to U(12). (6)

4. (a) State and prove Orbit Stabilizer Theorem. (6½)

(b) (i) Prove that $aH = H$ if and only if $a \in H$. (3)

(ii) Prove that $aH = bH$ or $aH \cap bH = \phi$. (3½)

(c) (i) Prove that order of U(n) is even when $n > 2$. (3)

(ii) Prove that a group of prime order is cyclic. (3½)

5. (a) Let H and K be subgroups of a finite group G and let

$$HK = \{hk \mid h \in H,\ k \in K\}$$

and

$$KH = \{kh \mid k \in K,\ h \in H\}.$$

Prove that HK is a group if and only if $HK = KH$. (6½)

(b) Let φ be a homomorphism from a group G to a group $\bar{G}$ and let g be an element of G. Prove that

(i) If $\varphi(g) = g'$, then $\varphi^{-1}(g') = \{x \in G | \varphi(x) = g'\} = g\text{Ker}\varphi$ (4)

(ii) If $|\text{Ker}\varphi| = n$, then φ is an $n - to - 1$ mapping from G onto $\varphi(G)$. (2½)

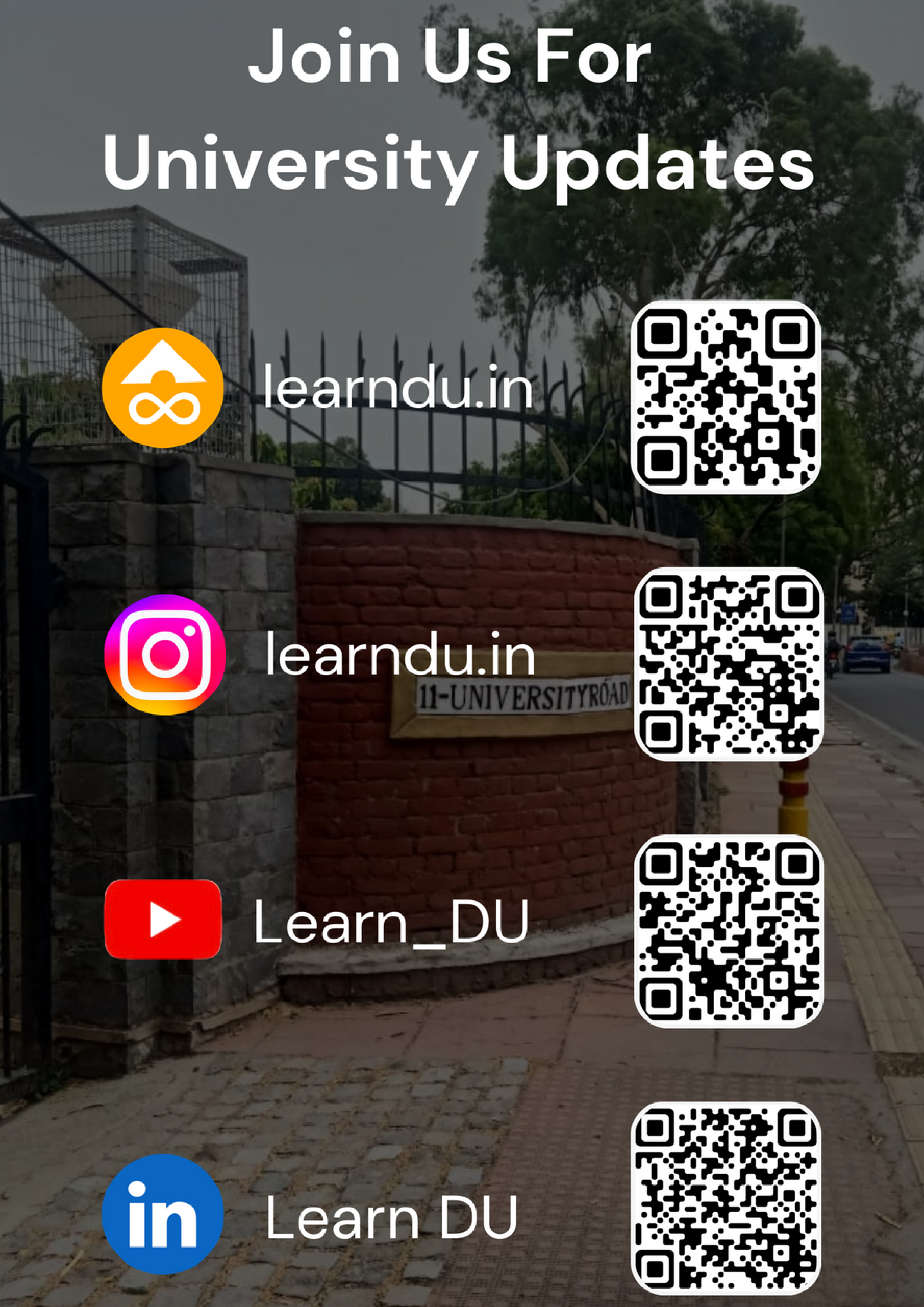
(c) (i) Prove that A_n is normal in S_n. (3½)

(ii) If G is a non-Abelian group of order p^3 (p is prime) and $Z(G) \neq \{e\}$, prove that $|Z(G)| = p$. (3)

6. (a) State and prove The First Isomorphism Theorem. (6½)

(b) Let G be a group and let Z(G) be the center of G. Prove that if $G/Z(G)$ is cyclic, then G is Abelian. (6½)

(c) Let $4Z = (0, \pm 4, \pm 8, \cdots\}$. Find $Z/4Z$. (6½)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