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Unique paper code : 2351301

Name of the course : B. Sc. (Hons) Mathematics

Name of the paper : Algebra II (Group Theory - I)

Semester : III

Duration : 3 Hours

Maximum marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of the question paper.
2. Attempt **any two parts** from each question.
3. All questions are compulsory.

1(a) (i) For any elements a and b from a group and any integer n prove that $(a^{-1}ba)^n = a^{-1}b^n a$.

(ii) Give an example of a non-cyclic group all of whose proper subgroups are cyclic.

(b) Define center of a group. Prove that the center of a group G is a subgroup of G.

(c) If $G = \langle a \rangle$ is a cyclic group of order n then prove that $G = \langle a^k \rangle$ *iff* g.c.d $(k, n) = 1$.

(6 × 2 = 12)

2(a) Let $G = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} \mid a \in \mathbb{R}, a \neq 0 \right\}$. Show that G is a group under matrix multiplication.

(b) (i) Let H be a non empty finite subset of a group G. Then prove that H is a subgroup of G if H is closed under the operation of G.

(ii) Let G be a group and let a be any element of G. Then prove that $\langle a \rangle$ is subgroup of G.

(c) (i) How many subgroup does Z_{30} have. List a generator for each of these.

(ii) Prove that $H = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \mid n \in \mathbb{Z} \right\}$ is a cyclic subgroup of $GL(2, \mathbb{R})$ (6 × 2 = 12)

3(a) Prove that the order of a permutation of a finite set written as a product of disjoint cycles, is the least common multiple of the lengths of the cycles.

(b) State and prove Lagrange's Theorem. Is the converse true? Justify your answer.

(c) Let G be a group and H a normal subgroup of G. Prove that the set $G/H = \{aH | a \in G\}$ is a group under the operation $(aH)(bH) = abH$. (6 × 2 = 12)

4. (a) Show that if H is a subgroup of S_n ($n \geq 2$) then either every member of H is an even permutation or exactly half of them are even.

(b) State and prove Fermat's Little Theorem.

(c)(i) Prove that a subgroup H of a group G is a normal subgroup of G if and only if

$ghg^{-1} \in H$ for all $g \in G$ and for all $h \in H$.

(ii) Suppose G is a group and $H = \{g^2 : g \in G\}$ is a subgroup of G. Prove that H is a normal subgroup of G. (6.5 × 2 = 13)

5. (a) Let G be a group and Z (G) be the centre of G. If G/Z (G) is cyclic then prove that G is Abelian.

(b) Show that any infinite cyclic group is isomorphic to (Z, +) the group of integers under addition.

(c) Let G be a group of permutation and {1,-1} be the multiplicative group. For each $\sigma \in G$, define a mapping

$\varphi : G \to \{1, -1\}$,

by

$$\varphi(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is an even ;} \\ -1 & \text{if } \sigma \text{ is an odd.} \end{cases}$$

Prove that φ is a group homomorphism. Also, find Ker φ. (6.5 × 2 = 13)

6. (a) Suppose that φ is an isomorphism from a group G onto a group G*. Prove that G is cyclic if and only if G* is cyclic. Hence show that Z, the group of integers under addition is not isomorphic to Q, the group of rationals under addition.

(b) If M and N are normal subgroups of a group G and $N \leq M$, prove that $(G/N) / (M/N) \approx G/M$.

(c) Let φ be a group homomorphism from G onto G* then prove that $G/\text{Ker } \varphi \approx G^*$. (6.5 × 2 = 13)

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