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S. No. of Question Paper : 90

(12)

Unique Paper Code : 32351301

Name of the Paper : Theory of Real Functions

Name of the Course : B.Sc. (Hons.) Mathematics

Semester : III

Duration : 3 Hours

Maximum Marks : 75

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt any *three* parts from each question

*All* questions are compulsory.

1. (a) Let  $A \subseteq \mathbb{R}$ ,  $f: A \rightarrow \mathbb{R}$  and  $c \in \mathbb{R}$  be a cluster point of  $A$ . Then prove that  $f$  can have only one limit at  $c$ . 5

(b) Use the  $\epsilon$ - $\delta$  definition of the limit to prove that

$$\lim_{x \rightarrow c} x^3 = c^3 \text{ for any } c \in \mathbb{R}. \quad 5$$

(c) State divergence criterion for limit of a function. Show

that  $\lim_{x \rightarrow 0} (x + \operatorname{sgn}(x))$  does not exist. 5



(d) Prove that :

$$(i) \quad \lim_{x \rightarrow 0^+} \frac{1}{x} = \infty$$

$$(ii) \quad \lim_{x \rightarrow \infty} \frac{1}{x^2} = 0.$$

5

(a) Let  $A \subseteq \mathbb{R}$ ,  $f, g, h : A \rightarrow \mathbb{R}$  and  $c \in \mathbb{R}$  be a cluster point of  $A$ . If  $f(x) \leq g(x) \leq h(x)$  for all  $x \in A$ ,  $x \neq c$  and if  $\lim_{x \rightarrow c} f(x) = L = \lim_{x \rightarrow c} h(x)$ , then prove that

$$\lim_{x \rightarrow c} g(x) = L.$$

5

(b) State and prove sequential criterion for continuity of a real valued function.

5

(c) Let the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$f(x) = \begin{cases} 2x & : \text{if } x \text{ is rational} \\ x+3 & : \text{if } x \text{ is irrational} \end{cases}$$

Find all the points at which  $f$  is continuous.

5

(d) Let  $x \rightarrow [x]$  denote the greatest integer function. Determine the points of continuity of the function  $f(x) = x - [x]$ ,

$x \in \mathbb{R}$ .

5

3. (a) Let  $f$  be a continuous real valued function defined on  $[a, b]$ . By assuming that  $f$  is a bounded function show that  $f$  attains its bounds on  $[a, b]$ . 5
- (b) State Bolzano's Intermediate value theorem and show that the function  $f(x) = xe^x - 2$  has a root  $c$  in the interval  $[0, 1]$ . 5
- (c) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous on  $\mathbb{R}$  and suppose that  $f(r) = 0$  for every rational numbers  $r$ . Show that  $f(x) = 0$  for all  $x \in \mathbb{R}$ . 5
- (d) Define uniform continuity of a function. Prove that if a function is continuous on a closed and bounded interval  $I$ , then it is uniformly continuous on  $I$ . 5
4. (a) Show that the function  $f(x) = 1/x^2$  is uniformly continuous on  $A = [0, \infty[$  but it is not uniformly continuous on  $B = ]0, \infty[$ . 5
- (b) Determine where the following function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is differentiable,  $f(x) = |x - 1| + |x + 1|$ . 5



- (c) Let  $f$  be defined on an interval  $I$  containing the point  $c$ . Then prove that  $f$  is differentiable at  $c$  if and only if there exists a function  $\phi$  on  $I$  that is continuous at  $c$  and satisfies  $f(x) - f(c) = \phi(c)(x - c)$  for all  $x \in I$ . In this case, we have  $\phi(c) = f'(c)$ . Using the above result find the function  $\phi$  for  $f(x) = x^3, x \in \mathbb{R}$  5
- (d) State and prove Mean Value Theorem. 5
5. (a) State Darboux's theorem. Suppose that  $f: [0, 2] \rightarrow \mathbb{R}$  is continuous on  $[0, 2]$  and differentiable on  $]0, 2[$  and that  $f(0) = 0, f(1) = 1, f(2) = 1$ . (i) Show that there exists  $c_1 \in (0, 1)$  such that  $f'(c_1) = 1$ . (ii) Show that there exists  $c_2 \in (1, 2)$  such that  $f'(c_2) = 0$ . (iii) Show that there exists  $c \in (0, 2)$  such that  $f'(c) = 1/10$ . 5
- (b) Let  $f: I \rightarrow \mathbb{R}$  be differentiable on the interval  $I$ . Then prove that  $f$  is increasing on  $I$  if and only if  $f'(x) \geq 0$  for all  $x \in I$ . 5
- (c) State Taylor's theorem. Use it to prove that  $1 - x^2/2 \leq \cos x$  for all  $x \in \mathbb{R}$ . 5
- (d) Find the Taylor series for  $e^x$  and state why it converges to  $e^x$  for all  $x \in \mathbb{R}$ . 5



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